Problem 1 Let:

$$L = \{b^n a^k b^\ell a^j c^m \mid \ell = n, \; j > 2, \; k = 0, \; n, k, \ell, j, m \geq 0\}$$

(a) Write a complete formal definition of a context-free grammar that generates L. If such a grammar does not exist, prove it.

Answer:

The template is $b^{2m} a^{j+3} c^m$,

$$m, j, m \geq 3$$

whence the grammar: $G = (V, \mathcal{L}, P, S)$

$V = \{S, K, B, A\}, \quad \mathcal{L} = \{a, b, c\},$

$P: \quad S \to BAaaaK$

$B \to bbB \mid \lambda$

$A \to aA \mid \lambda$

$K \to cK \mid \lambda$

(b) Write a regular expression that defines L. If such a regular expression does not exist, prove it.

Answer:

$(bb)^* \, a^* aaa \, c^*$

Problem 2 Let:

LAST NAME: _____

FIRST NAME: Solution

$$L = \{c^n b^k c^l b^j a^m \mid k = l = m, \ j = 0, \ n, k, l, j, m \geq 0\}$$

(a) Write a complete formal definition of a context-free grammar that generates L. If such a grammar does not exist, prove it.

Answer: The template is $c^n B^l c^l a^l$, $n, l \geq 0$ and the grammar does not exist since L is not context-free. To prove it, assume the opposite. Observe that every string has the property:

(*) $\#a's = \#'bs = \#c's$ after the last b.

Let k be the constant of the Pumping Lemma. Select the word, $w_0 = b^l c^l a^l$, where l is selected so that $l > k$. In any pumping decomposition, the pumping window is shorter than k and shorter than l, hence at least one letter in never pumped and at least one is violating property (*).

(b) Draw a state transition gfraph of a finite automaton that accept L. If such an automaton does not exist, prove it.

Answer:

Impossible, since L is not regular. If L was regular, it would be context-free, since all regular languages are context-free. By the result of part (a), L is not context-free. Hence, L cannot be regular.

Problem 1   Let:

$$L = \{b^n a^k b^\ell a^j c^m \mid \ell = n,\ j > 2,\ k = 0,\ n, k, \ell, j, m \geq 0\}$$

(a) Write a complete formal definition of a context-free grammar that generates L. If such a grammar does not exist, prove it.

Answer:

The template is $b^{2m} a^{j+3} c^m$,

$$m, j, m \geq 3$$

whence the grammar: $G = (V, \Sigma, P, S)$

$V = \{S, K, B, A\}$, $\Sigma = \{a, b, c\}$,

P:

$S \rightarrow BAaaaK$

$B \rightarrow bbB \mid \wedge$

$A \rightarrow aA \mid \wedge$

$K \rightarrow cK \mid \wedge$

(b) Write a regular expression that defines L. If such a regular expression does not exist, prove it.

Answer:

$$(bb)^* \, a^* aaa \, c^*$$

Problem 2   Let:

LAST NAME:

FIRST NAME: Solution

$$L = \{c^n b^k c^\ell b^j a^m \mid k = \ell = m,\ j = 0,\ n, k, \ell, j, m \geq 0\}$$

(a) Write a complete formal definition of a context-free grammar that generates L. If such a grammar does not exist, prove it.

Answer: The template is $c^n b^\ell c^\ell a^\ell$, $n, \ell \geq 0$
and the grammar does not exist since L is not context-free. To prove it, assume the opposite.
Observe that every string has the property:

(⋆) $\boxed{\#a's = \#'bs = \#c's \text{ after the last } b.}$

Let v be the constant of the Pumping Lemma.
Select the word, $w_0 = b^\ell c^\ell a^\ell$, where ℓ is
selected so that $\ell > k$. In any pumping
decomposition, the pumping window is
shorter than k and shorter than ℓ, hence
at least one letter is never pumped and
at least one is
violating property (⋆).

(b) Draw a state transition gfraph of a finite automaton that accept L. If such an automaton does not exist, prove it.

Answer:

Impossible, since L is not regular.
If L was regular, it would be
context-free, since all regular languages
are context-free. By the result of
part (a), L is not context-free.
Hence, L cannot be regular.

Problem 3 Let:

LAST NAME: _____

FIRST NAME: *Solution*

$$L = \{a^n c^k a^\ell c^j b^m \mid j = \ell = n,\ m > 1,\ k = 0,\ n, k, \ell, j, m \geq 0\}$$

(a) Write a complete formal definition of a context-free grammar that generates L. If such a grammar does not exist, prove it.

Answer:

The template is: $a^{2n} c^n bbb^m$,

$n, m \geq 0$

whence the grammar: $G = (V, \Sigma, P, S)$

$V = \{S, A, B\}$, $\Sigma = \{a, b, c\}$

P: $S \to AbbB$

$A \to aaAc \mid \lambda$

$B \to bB \mid \lambda$

(b) Draw a state transition graph of a finite automaton that accepts L. If such an automaton does not exist, prove it.

Answer: Impossible since L is not regular. Assume the opposite. Observe that every string of L has the property:

$[\#a's = \text{twice the } \# c's.]$ $(*)$

Let κ be the constant of the Pumping Lemma. Select a word $w_0 = a^{2n} c^n bb^{m+2}$ where n is selected so that $n > \kappa$. In any pumping decomposition, the pumping window is shorter than κ and shorter than n and thus consists of a's only. Pumping up once violates $(*)$.

Problem 4 Let L be the language accepted by the pushdown automaton: $M = (Q, \Sigma, \Gamma, \delta, q, F)$ where: $Q = \{q, p\}$, $\Sigma = \{a, b, c, d, e\}$, $\Gamma = \{A, E, M, X\}$, $F = \{p\}$ and the transition function δ is defined as follows:

$$[q, e, \lambda, p, EXAM]$$
$$[p, a, A, p, \lambda]$$
$$[p, a, E, p, \lambda]$$
$$[p, b, M, p, \lambda]$$
$$[p, c, X, p, \lambda]$$
$$[p, d, \lambda, p, \lambda]$$

(Recall that M is defined so as to accept by final state and empty stack. Furthermore, if an arbitrary stack string, say $X_1 \ldots X_n \in \Gamma^*$ where $n \geq 2$, is pushed on the stack by an individual transition, then the leftmost symbol X_1 is pushed first, while the rightmost symbol X_n is pushed last.)
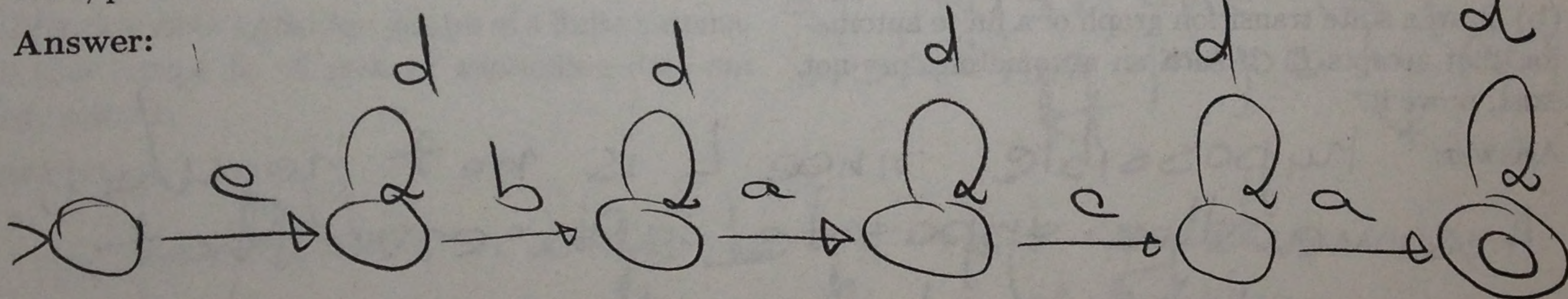
(a) List 6 distinct strings that belong to L. If this is impossible, state it and explain why.

Answer: Advice: L is

$$e\, d^\star\, b\, d^*\, a\, d^*\, c\, d^*\, a\, d^*$$

(b) Draw a state transition graph of a finite automaton that accepts L. If such an automaton does not exist, prove it.

Answer:

LAST NAME: _____

FIRST NAME: *Solution*

(c) What is the cardinality of the set L? If it is finite, state the exact number; if it is infinite, state whether it is countable or uncountable.

Answer:

L is infinite and countable.

(d) What is the cardinality of the set $\mathcal{P}(L)$ (the set of subsets of L?) If it is finite, state the exact number; if it is infinite, state whether it is countable or uncountable.

Answer:

$\mathcal{P}(L)$ is infinite and uncountable.

Problem 5   Let L be the language accepted by the pushdown automaton: $M = (Q, \Sigma, \Gamma, \delta, q, F)$ where: $Q = \{q, p\}$, $\Sigma = \{a, b, c, d, e\}$; $\Gamma = \{A, E, M, X\}$, $F = \{p\}$ and the transition function δ is defined as follows:

$$[q, e, \lambda, q, EX]$$
$$[q, e, \lambda, q, AM]$$
$$[q, \lambda, \lambda, p, \lambda]$$
$$[p, b, E, p, \lambda]$$
$$[p, a, X, p, \lambda]$$
$$[p, c, A, p, \lambda]$$
$$[p, d, M, p, \lambda]$$

(Recall that M is defined so as to accept by final state and empty stack. Furthermore, if an arbitrary stack string, say $X_1 \ldots X_n \in \Gamma^*$ where $n \geq 2$, is pushed on the stack by an individual transition, then the leftmost symbol X_1 is pushed first, while the rightmost symbol X_n is pushed last.)

(a) List 6 distinct strings that belong to L. If this is impossible, state it and explain why.

Answer:

λ, eab, edc,
eeabdc, eeabab,
eedcab

(b) Write a complete formal definition of a context-free grammar that generates L. If such a grammar does not exist, prove it.

Answer:

$G = (V, \mathcal{L}, P, S)$

$V = \{S\}$, $\mathcal{L} = \{a, b, c, d\}$

P:

$S \rightarrow eS(ab)eS(dc) \mid \lambda$

LAST NAME: _____

FIRST NAME: Solution

(c) State one trivial property of the language L, such that a^*b^* does not have this property. Explain carefully why this property is trivial, and prove that L indeed has it, while a^*b^* does not. If such a property does not exist, state it, and explain why it is so.

Answer:

Impossible. If this property existed, it would be true for L and false for a^*b^* and by definition could not be trivial (which always assumes the same value.)

(d) State one non-trivial property of the language L, such that a^*b^* does not have this property. Explain carefully why this property is non-trivial, and prove that L indeed has it, while a^*b^* does not. If such a property does not exist, state it, and explain why it is so.

Answer:

Such a property is

"every nonempty string begins with e"

L has this property, by its grammar. a^*b^* does not have it, since no string in a^*b^* contains e. Hence, the property has different values for two languages, and is by definition non-trivial.

a*b*c*, d*

Problem 6 Consider the following Turing machine: $M = (Q, \Sigma, \Gamma, \delta, q, F)$ such that:
$Q = \{q, r, s, p, v, t, z, x, y\}$;
$\Sigma = \{0, 1\}$; $\Gamma = \{B, 0, 1\}$; $F = \{x\}$; and δ is defined by the following transition set:

$$[q, 0, r, 0, R] \quad [v, 0, x, 0, L]$$
$$[r, 1, s, 1, R] \quad [v, 1, z, 1, L]$$
$$[s, 0, t, 0, R]$$

$$[z, 0, y, 0, L]$$
$$[t, 0, p, 0, R] \quad [z, 1, x, 1, L]$$
$$[t, 1, p, 1, R]$$

$$[y, 0, y, 0, R]$$
$$[p, 0, p, 0, R] \quad [y, 1, y, 1, R]$$
$$[p, 1, p, 1, R] \quad [y, B, y, B, R]$$
$$[p, B, v, B, L]$$

(Assume that M is defined so as to have an one-way infinite tape (infinite to the right only.) B is the designated blank symbol. M accepts by final state.)

Let L be the set of strings on which M *diverges*.

(a) List 6 distinct strings that belong to L. If this is impossible, state it and explain why.

Answer: <u>Advice</u>

See part (b).

(b) Write a regular expression that defines L. If such a regular expression does not exist, prove it.

Answer:

$$010 \, (0 \cup 1)^* \, 01 \cup 0101$$

(c) Dangerous Professor has told her students to write a program that operates as follows:

INPUT: String w over $\{0, 1\}$.

OUTPUT: **yes** if w represents a Turing Machine which halts exactly when the Turing Machine M (defined at the beginning of this problem) diverges; **no** otherwise.

Explain the algorithm that should be employed by this program, or state that it does not exist, and prove it.

Answer: Impossible.

If this algorithm existed, it would decide the set of TMs whose languages have the property

"accepts by halting the set of strings on which M diverges", or in short, it would decide TMs whose languages $= L$. Since property ① is non-trivial, by Rice's theorem, the construction is impossible.

[properties:
① "accepts by halting the set of strings on which M diverges", or in short, whose languages $= L$. But say ①
L has this property but say ② does not, property is non-trivial, and by Rice's theorem, the construction is impossible.]

Problem 7 Consider the following Turing machine: $M = (Q, \Sigma, \Gamma, \delta, q)$ such that:
$Q = \{q, p, v, z, x\}$;
$\Sigma = \{0, 1\}$; $\Gamma = \{B, 0, 1, N\}$; $F = \{x\}$; and δ is defined by the following transition set:

$$[q, 0, p, N, R] \quad [v, 1, v, 1, L]$$
$$[q, 1, q, 1, R] \quad [v, 0, x, 0, R]$$
$$[q, B, q, B, R] \quad [v, N, z, 0, R]$$

$$[p, 0, p, 0, R]$$
$$[p, 1, p, 1, R]$$
$$[p, B, v, B, L]$$

(Assume that M is defined so as to have an one-way infinite tape (infinite to the right only.) B is the designated blank symbol. M accepts by final state.)

Let L be the set of string which M rejects.

(a) List 6 distinct strings that belong to L. If this is impossible, state it and explain why.

Answer: <u>Advice</u>

See part (b)

(b) Write a regular expression that defines L. If such a regular expression does not exist, prove it.

Answer:

$$1^* 0 1^*$$

FIRST NAME: Solution

(c) Dangerous Professor has told her students to write a program that operates as follows:

INPUT: String w over $\{0, 1\}$.

OUTPUT: **yes** if w represents a Turing Machine that accepts exactly those strings which the Turing Machine M (defined at the beginning of this problem) rejects;

no otherwise.

Explain the algorithm that should be employed by this program, or state that it does not exist and prove it.

Answer:

Impossible. If this algorithm existed it would decide the set of TMs whose languages have the nontrivial property

" is equal to L".

Since the property is true for L but false say for $\emptyset$, it is nontrivial and by Rice's theorem, the construction is impossible.

Problem 8   Consider the following Turing machine: $M = (Q, \Sigma, \Gamma, \delta, q)$ such that:
$Q = \{q, p, v, z, x\}$;
$\Sigma = \{0, 1\}$; $\Gamma = \{B, 0, 1\}$; $F = \{x\}$; and δ is defined by the following transition set:

$$[q, 0, q, 0, R] \quad [v, 0, x, 0, R]$$
$$[q, 1, p, 1, R] \quad [v, 1, z, 1, R]$$
$$[q, B, q, B, R]$$

$$[p, 1, q, 1, R]$$
$$[p, 0, p, 0, R]$$
$$[p, B, v, B, L]$$

(Assume that M is defined so as to have an one-way infinite tape (infinite to the right only.) B is the designated blank symbol. M accepts by final state.)

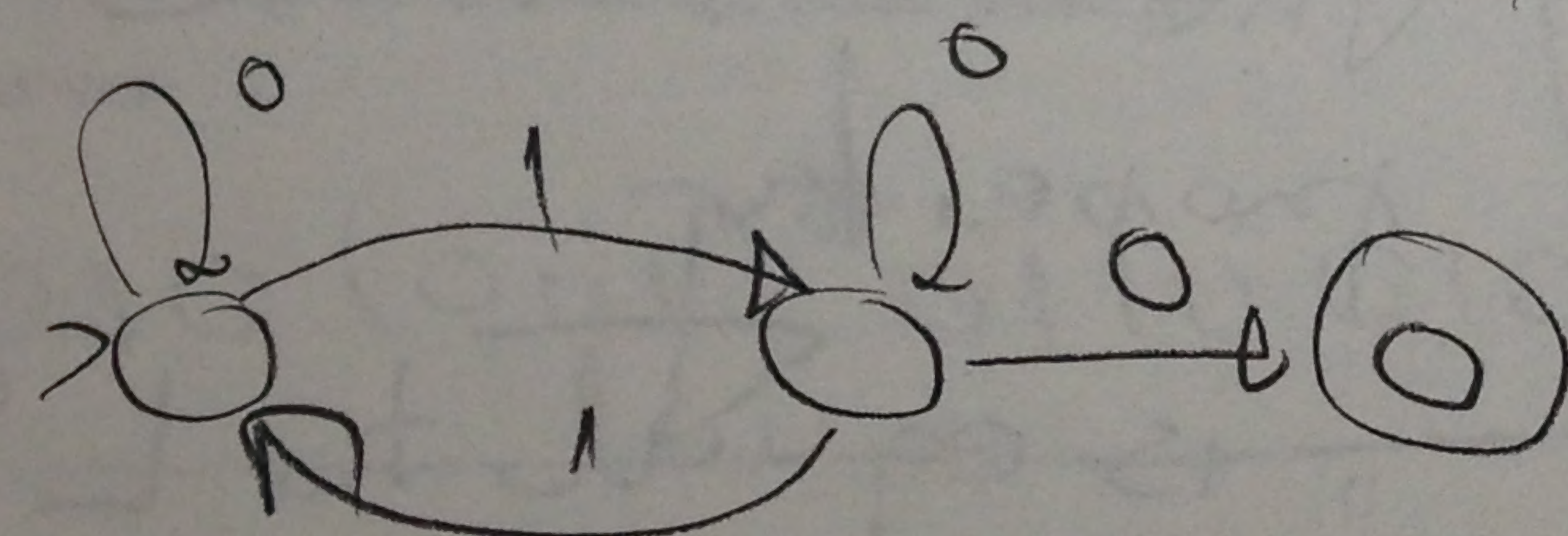Let L be the set of string which M accepts.

(a) List 6 distinct strings that belong to L. If this is impossible, state it and explain why.

Answer: <u>Advice:</u>

contains an odd
1's and ends with 0.

(b) Draw a state transition graph of a finite automaton that accepts L. If such an automaton does not exist, prove it.

Answer:

LAST NAME: _____

FIRST NAME: Solution

(c) Dangerous Professor has told her students to write a program that operates as follows:

INPUT: String w over $\{0, 1\}$.

OUTPUT: **yes** if w is a string such that the Turing Machine M (defined at the beginning of this problem) accepts w;
no otherwise.

Explain the algorithm that should be employed by this program, or state that it does not exist and prove it.

Answer:

Convert the finite automaton obtained in the answer to part (b) to a deterministic equivalent, simulate this deterministic automaton, and decide exactly as it does.

Problem 9 Let L be the set of all strings over the alphabet $\{a, b, c\}$ which satisfy all of the following properties.

LAST NAME: _____

FIRST NAME: _Solution_

- if the string begins with a, then it contains an odd number of a's.

- if the string begins with b, then all of the following conditions hold:

  1. the string ends with b;
  2. the string has an odd length;
  3. the middle symbol is equal to the last symbol;

- if the the string begins with c, then both of the following conditions hold:

  1. the string has an even length;
  2. the string is a palindrome;

Write a complete formal definition of a context-free grammar that generates the language L. If such a grammar does not exist, state it and explain why.

Answer:

$$G = (V, \Sigma, P, S)$$

$$\Sigma = \{a, b, c\}, \quad V = \{S, A, D, E, B, M, Z, K, L\}$$

$$P: \quad S \to A \mid B \mid K$$

$$A \to aD$$
$$D \to aE \mid bD \mid cD \mid \lambda$$
$$E \to aD \mid bE \mid cE$$
$$B \to bMb \mid b$$
$$M \to ZMZ \mid b$$
$$Z \to a \mid b \mid c$$

$$K \to cLc$$
$$L \to aLa \mid bLb \mid cLc \mid \lambda$$